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COMPACT LIE GROUPS AND THE STABLE HOMOTOPY OF SPHERES

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§1. INTRODUCTION

A COMPACT Lie group G of dimension n determines by the Pontrjagin–Thom construction an element in the stable homotopy group of spheres $\pi_n{}^S$. More precisely a basis of the Lie algebra of G defines by left translation a trivialization of the tangent bundle of G and hence also a trivialization of the stable normal bundle (unique up to homotopy) to which one applies the Pontrjagin–Thom construction. The resulting element of $\pi_n{}^S$ depends only on the orientation of the basis and we shall denote it by $[G, \alpha, \mathscr{L}]$, where α is the orientation of G and $\mathscr{L}$ indicates that we have used left translation: replacing this by right translation we get another element $[G, \alpha, \mathscr{R}]$.

For $G = S^1$ or S^3 the elements we get in this way are easily identified with the generators of $\pi_1{}^S$ and $\pi_3{}^S$. For other Lie groups it is by no means clear whether we get interesting elements of $\pi_n{}^S$. Now the simplest and best understood part of $\pi_n{}^S$ is the image of the J-homomorphism

$$\pi_n(SO) \to \pi_n{}^S$$

(where $SO = SO(n + k)$ for large k). According to the results of Adams [1] and Quillen [11] the image of J is a *direct summand* of $\pi_n{}^S$ and, when $n \equiv 3 \bmod 4$, the projection of $\pi_n{}^S$ onto Im J is given essentially by the Adams e-invariant:

$$e\colon \pi_n{}^S \to Q/Z.$$

The purpose of this note is to dispose of the e-invariant for most Lie groups by proving

THEOREM. *Let G be a non-abelian compact connected Lie group of rank >1 and of dimension $4k - 1$. Assume further that the adjoint representation of G lifts to spin, then*

$$e[G, \alpha, \mathscr{L}] = 0.$$

In particular this holds for simply-connected groups.

Remarks

(1) Examples of Lie groups of the right dimension to which this theorem applies are the special unitary groups $SU(2n)$ and the Spinor groups Spin $(8n + 3)$, Spin $(8n + 6)$.

(2) The assumption that G is non-abelian excludes the 3-dimensional torus $S^1 \times S^1 \times S^1$ which represents the element $\eta^3 \in \pi_3{}^S$ with e-invariant $\frac{1}{2}$ ($\eta \in \pi_1{}^S$ is the generator represented by S^1).

(3) The assumption about the adjoint representation is in fact superfluous but to prove this requires more sophisticated techniques. The problem is that the e-invariant is not multiplicative for finite coverings. A general treatment of this question using the methods of [6] will be given elsewhere and is commented on in §4. Our present methods will however deal with the p-primary part of the e-invariant for *odd* primes p.

(4) One might be tempted to think that $[G, \alpha, \mathscr{L}]$ is always zero for rank $G > 1$ but this is not so. A careful study of simple groups of rank two [13] shows that they give rise to non-zero elements in $\pi_n{}^S$. By our theorem these elements are not contained in Im J. It seems likely that for groups of higher rank one will need increasingly complicated Toda brackets to represent the elements they define in $\pi_n{}^S$.

For a framed manifold X of dimension $4k - 1$ the e-invariant may be computed in the following way. Using the fact that the spin cobordism group is zero in these dimensions [14] we can express X as the boundary of a spin manifold Y of dimension $4k$, the induced spin structure on X being compatible with the framing. The tangent bundle of Y being trivialized over X we can define the Pontrjagin classes p_i as relative classes in $H^*(Y, X)$ and hence evaluate the $\hat{A}_k$-polynomial of Hirzebruch on the fundamental cycle of Y. In other words the $\hat{A}$-genus $\hat{A}(Y)$ is defined: it is a rational number and

$$\begin{aligned} e[X] &= \hat{A}(Y) \bmod Z, \qquad k \text{ even} \\ &= \tfrac{1}{2}\hat{A}(Y) \bmod Z, \qquad k \text{ odd}. \end{aligned} \tag{1.1}$$

This is independent of the choice of Y in view of the integrality theorems for the $\hat{A}$-genus of closed spin manifolds [4]. This description of the e-invariant is essentially due to Conner and Floyd. In [9] there is a proof due to Landweber proving the analogous result for the slightly weaker complex e-invariant, but exactly the same type of proof (replacing complex K-theory by real K-theory, complex cobordism by spin cobordism, and the Todd polynomial by $\hat{A}$) yields (1.1).

For a Lie group G satisfying the hypotheses of our theorem it is easy to find a suitable Y, with $\partial Y = G$, using a fibration $G \to G/S^3$ (see §3). However the cohomological calculation needed to evaluate $\hat{A}(Y)$ would be somewhat tiresome because one has to pay due regard to the framing on the boundary. Instead we resort to a variety of stratagems which avoid a direct calculation. As one would expect, the 2-primary part of the e-invariant presents the most difficulties so we begin by establishing in §2 a weaker result, namely that $2e[G, \alpha, \mathscr{L}] = 0$. In §3 we remove the factor 2 by using the main result of [5], which enables us to lift the e-invariant from Q/Z to Q.

As we have already mentioned S^3 gives rise to a generator of $\pi_3{}^S \cong Z_{24}$ so that $e[S^3] = \pm\frac{1}{24}$. For $SO(3)$ we shall show that $2e[SO(3)] = \pm\frac{1}{6}$, exemplifying the non-multiplicative behaviour of e. In fact it can be shown that $e[SO(3)] = \pm\frac{1}{12}$.

Perhaps we should also point out that the invariant $d_{\mathbf{R}}\colon \pi_n{}^S \to Z_2$, for $n \equiv 1$ or $2 \bmod 8$, which is the Hurewicz map in real K-theory (see [1]) also vanishes for any *non-abelian* compact Lie group G. In fact on such a group there is a bi-invariant metric of *positive scalar* *curvature*. Hence by a result of Lichnerowicz [10] there are no harmonic spinors (except zero) on G. But by the results of [8]

$$\begin{aligned} d_{\mathbf{R}}[X] &= \dim_{\mathbf{R}} H \qquad \bmod 2, \quad \text{if} \quad \dim X \equiv 1 \bmod 8 \\ &= \dim_C H \qquad \bmod 2 \quad \text{if} \quad \dim X \equiv 2 \bmod 8 \end{aligned}$$

where H is the space of harmonic spinors on X (for some metric). Thus $d_{\mathbf{R}}[G] = 0$. Note that, for the abelian groups S^1 or $S^1 \times S^1$, d_R is non-zero.

The computation of the real Adams e-invariant for Lie groups of dimension $8k$ or $8k + 1$ seems to be more difficult and does not fall within the scope of our methods.

§2. A PARTIAL RESULT

Observe first that

$$[G, \alpha, \mathscr{R}] = [G, -\alpha, \mathscr{L}] = -[G, \alpha, \mathscr{L}]. \tag{2.1}$$

The first equality in (2.1) comes by applying inversion in G which interchanges $\mathscr{L}$ and $\mathscr{R}$ while reversing orientation (since dim G is odd). The second equality is a quite general and trivial result expressing the fact that the cylinder $G \times I$ has the two ends as boundary (with correct orientation and framing). From (2.1) we deduce

$$2e[G, \alpha, \mathscr{L}] = e[G, \alpha, \mathscr{L}] - e[G, \alpha, \mathscr{R}]. \tag{2.2}$$

Suppose now that $\partial Y = G$, then $Y' = Y \cup (G \times I)$ also has G as boundary. Using Y to compute $e[G, \alpha, \mathscr{R}]$ from formula (1.1) and Y' to compute $e[G, \alpha, \mathscr{L}]$ we see that the right hand side of (2.2) can be computed from the cylinder $G \times I$ with trivializations $\mathscr{L}$ and $\mathscr{R}$ at the two ends. In this we have to be careful to check that the spin structures defined by $\mathscr{L}$ and $\mathscr{R}$ coincide and this is precisely where we need our restriction on the adjoint representation, which measures the difference between $\mathscr{L}$ and $\mathscr{R}$. Using the suspension isomorphism $H^{4k}(G \times I, G \times \partial I) \cong H^{4k-1}(G)$ we obtain therefore

$$2e[G, \alpha, \mathscr{L}] = (\mathrm{ad}^* \sigma \hat{A}_{4k})[G] \tag{2.3}$$

for k even and a similar formula with a factor $\frac{1}{2}$ for k odd. Here

$$\mathrm{ad}^*\colon H^*(SO(4k-1)) \to H^*(G) \tag{2.4}$$

is induced by the adjoint representation and

$$\sigma\colon H^{4k}(BSO(4k-1)) \to H^{4k-1}(SO(4k-1))$$

is induced by the map $H \times I/H \times \partial I \to BH$ for $H = SO(4k-1)$. Note that, being induced by a group homomorphism, ad* is a homomorphism of Hopf algebras and hence sends primitive† elements to primitive elements. On the other hand every element in the image of σ is primitive and so $\mathrm{ad}^* \sigma \hat{A}_{4k}$ is primitive in $H^*(G)$. But $H^*(G)$ is an exterior algebra on primitive generators and so, for rank $G > 1$, we can have no (non-zero) primitive element in the highest dimension. Thus

$$2e[G, \alpha, \mathscr{L}] = 0$$

as asserted.

† Recall that x is primitive if $\Delta x = x \otimes 1 + 1 \otimes x$ where Δ is the diagonal map of the Hopf algebra.

Note. For $G = S^3$ the roots are $\pm 2x$, so $\text{ad}^* p_1 = 4x^2 \in H^4(B_G)$. Since $\hat{A}_1 = -\frac{p_1}{24}$ and $\sigma x^2[S^3] = 1$, formula (2.3) for odd k, gives

$$2e[S^3, \alpha, \mathscr{L}] = -\frac{1}{12}$$

where α denotes the standard orientation (induced from the unit ball in $\mathbf{C}^2$). In fact instead of comparing $\mathscr{L}$ with $\mathscr{R}$ we can here compare $\mathscr{L}$ with the trivial framing of $\mathbf{C}^2$ and a similar computation then gives

$$e[S^3, \alpha, \mathscr{L}] = -\frac{1}{24},$$

identifying $[S^3, \alpha, \mathscr{L}]$ with one of the generators of $\pi_3{}^S$.

For $SO(3)$ we cannot use this method because ad does not lift to spin. Instead we observe that $SO(3)$ is the boundary of a complex 2-manifold Y, namely the unit cotangent bundle of $S^2 = P_1(\mathbf{C})$. The outward normal to ∂Y together with the framing $\mathscr{R}$ of $SO(3)$ gives a complex framing along ∂Y and hence relative Chern classes c_1, c_2 are defined. Since $H^2(Y, \partial Y)$ restricts injectively to H^2 of the zero section we see that $c_1 = 0$ (the tangent and cotangent bundles of P_1 having opposite Chern classes). By the Hopf theorem for manifolds with boundary

$$c_2 = \text{Euler class} = 2g$$

where g is the positive generator of $H^4(Y, \partial Y)$. Thus the relative *Todd genus* $T(Y)$ is given by

$$T(Y) = \frac{c_1{}^2 + c_2}{12}[Y] = \frac{1}{6}.$$

Now as mentioned in §1 the complex e-invariant $e_{\mathbf{C}}(\partial Y)$ is computed, according to [9], by the relative Todd class of Y. Since, in dimension 4, $e_{\mathbf{C}} = 2e$ we deduce

$$2e[SO(3), \alpha, \mathscr{R}] = \frac{1}{6}$$

where α is the orientation induced by taking S^3 as the boundary of the unit ball in $\mathbf{C}^2$.

The above calculations exemplify the non-multiplicative behaviour of the e-invariant for finite coverings. However a quite general argument shows that the p-primary part of the e-invariant is multiplicative for coverings of degree *prime to p*. By passing to a double covering of G we can always lift ad to spin and hence the odd-primary part of $e[G, \alpha, \mathscr{L}]$ vanishes for all G of rank > 1. Alternatively, to compute the odd-primary part of the e-invariant we can according to [14, p. 215] use the Hirzebruch L-polynomial instead of $\hat{A}$. Oriented cobordism now replaces spin cobordism so we can repeat our argument above without the spin restriction to deduce once more that the odd-primary part of $e[G, \alpha, \mathscr{L}]$ vanishes.

§3. A RATIONAL e-INVARIANT

We shall now show how to eliminate the factor 2 by defining, in appropriate circumstances an invariant E taking values in Q which gives e when we reduce modulo Z. The argument of §2 will then prove $2E[G, \alpha, \mathscr{L}] = 0$ and now we can divide by 2 and then reduce modulo Z to obtain $e[G, \alpha, \mathscr{L}] = 0$.

We recall that (1.1) defines an invariant in Q/Z because the $\hat{A}$-genus of a closed spin-manifold is an integer (an even integer if k is odd). If we can work with a restricted class of manifolds for which the $\hat{A}$-genus is actually *zero* then we can clearly define an invariant with values in Q. Now the main result of [5] asserts that *a closed spin-manifold Y which admits a non-trivial action of the circle group satisfies $\hat{A}(Y) = 0$.* Suppose therefore that X is a framed $(4k-1)$-manifold with a non-trivial circle action and assume that we can find spin manifolds Y with circle action such that $\partial Y = X$ (the circle actions and spin structures on Y being compatible with those on X). Then $\hat{A}(Y)$ (or $\frac{1}{2}\hat{A}(Y)$ if k is odd) defines an invariant

$$E[X] \in Q$$

independent of the choice of Y. Note that the circle action as well as the framing is considered part of the data of X, but no compatibility is required between the circle action and the framing. Clearly, after reducing modulo Z, $E[X]$ becomes $e[X]$.

To apply this to our Lie group G we must first decide on the circle action and then we must show that G bounds in our new sense, that is including the circle action. For this we need the following lemma:

LEMMA 3.1. *Let G be a compact non-abelian Lie group such that ad lifts to Spin. Then G contains a subgroup isomorphic to S^3.*

Proof. The hypothesis that ad lifts to Spin translates algebraically into the assertion that ρ, half the sum of the positive roots, is an integral weight (see for example [2; 5.56]). Let† α be a root of G, X_α the corresponding root-plane in the Lie algebra and G_α the subgroup generated by X_α. Then G_α is locally isomorphic to S^3, so either $G_\alpha \cong S^3$ or $G_\alpha \cong S^3/\pm 1$. To ensure that we have the first alternative we pick α to be a *simple* root and we let ω be the corresponding generator of the Weyl group of G. Then ω defines a reflection on $L(T)$, the Lie algebra of the maximal torus of G whose (-1)-eigenspace is $L(T_\alpha)$ where $T_\alpha = T \cap G_\alpha$ is the maximal torus (circle) of G_α. Hence for any linear form f on $L(T)$, the restriction $f_\alpha = f | L(T_\alpha)$ satisfies

$$f_\alpha = \frac{f_\alpha - \omega(f_\alpha)}{2}.$$

In particular taking $f = \rho$, and using the fact [12; Chapter V §9] that

$$\omega(\rho) = \rho - \alpha$$

we see that $\rho_\alpha = \frac{1}{2}\alpha_\alpha$. But α_α, the restriction of α to $L(T_\alpha)$, is just the unique root (up to sign) of G_α. Since half of this is an integral weight it follows that $G_\alpha \cong S^3$.

We now take the subgroup S^3 given by the lemma and we consider it as acting on G by *conjugation*. Restricting to any fixed $S^1 \subset S^3$ gives a circle action. The unit disc bundle of the fibration $G \to G/S^3$ gives us a $4k$-dimensional spin-manifold Y with $\partial Y = G$. Conjugation

† For the following standard facts we refer to [2] or [12].

preserves this fibration so that S^1 acts also on Y. Hence our refined invariant $E[G, \alpha, \mathscr{L}]$ is well-defined.

Since inversion in G is compatible with conjugation it follows that

$$E[G, \alpha, \mathscr{R}] = E[G, -\alpha, \mathscr{L}].$$

Moreover, by considering $G \times I$, we see that

$$E[G, -\alpha, \mathscr{L}] = -E[G, \alpha, \mathscr{L}]$$

and so (2.2) holds with e replaced by E. The remainder of the argument in §2 applies without change so that we finally obtain

$$2E[G, \alpha, \mathscr{L}] = 0$$

for rank $G > 1$. Since E takes values in Q this implies $E = 0$, hence also $e = 0$ completing the proof of our theorem.

§4. THE e-INVARIANT AND FINITE COVERINGS

We conclude with a few comments about the case of general non-abelian G. In view of our theorem the essential point is to study the behaviour of the e-invariant for double coverings. Suppose more generally that $\pi: \tilde{X} \to X$ is a finite covering of degree d, that X is framed and that $\tilde{X}$ is given the induced framing by π. The computations made in §2, showing how e is altered by a change of framing, prove at once that

$$\beta = e[\tilde{X}] - de[X] \tag{4.1}$$

is *independent of the choice of framing on* X. In fact β can be defined only assuming that X is a spin-manifold. For this we choose spin-manifolds Y, $\tilde{Y}$ with $\partial Y = X$, $\partial \tilde{Y} = \tilde{X}$ and connections for their tangent bundles which agree, via π, near their boundaries. Now put†

$$\beta = \int_{\tilde{Y}} \hat{A}_k(p) - d \int_Y \hat{A}_k(p) \tag{4.2}$$

where $\hat{A}_k(p)$ denotes the $4k$-form obtained by using the Pontrjagin forms of the connections. As before this is easily seen to be independent of the connections. For a framed manifold X, choosing the flat connection given by the framing recovers definition (3.2).

Now one can define an invariant for free actions of finite groups on spin-manifolds as in [7; §7]. It will be shown elsewhere, using the methods of [6], that β can be computed from the invariants in [7], and that for Lie groups of rank > 1 we always get zero. This is not quite enough because of the factor d in (4.1). However we can again refine our invariants to lie in characteristic zero (see for example [5; §4]) and this finally removes the factor d. For the covering $S^3 \to SO(3)$ the computations in [3; §8] show that $\beta = -\dfrac{1}{8}$ and hence that $e[SO(3). \alpha, \mathscr{R}] = \dfrac{1}{12}$.

† For odd k we use $\frac{1}{2}\hat{A}$.

The behaviour of the odd-primary part of the e-invariant under finite coverings can be dealt with more easily. Although this is not relevant to our problem with Lie groups (only the 2-primary part being left open at this stage) it is of some general interest and so we conclude with a brief treatment.

As noted before we can replace $\hat{A}$ by L if we ignore the 2-primary information and the spin restrictions can then be dropped everywhere. Instead of (4.2) we therefore consider

$$\gamma(\tilde{X}) = \int_{\tilde{Y}} L_k(p) - d\int_Y L_k(p) \tag{4.3}$$

as a measure of deviation of the e-invariant from multiplicativity. We can lift γ from Q/Z back to Q by defining

$$\Gamma(\tilde{X}) = \int_{\tilde{Y}} L_k(p) - \text{Sign } \tilde{Y} - d\left\{\int_Y L_k(p) - \text{Sign } Y\right\}. \tag{4.4}$$

The additivity of the signature, as explained in [7; §7], shows that Γ is well-defined, independent of the choice of Y, $\tilde{Y}$. Now assume for simplicity that $\tilde{X} \to X$ is a regular covering with group G (of order d), then for some integer N we have $N\tilde{X} = \partial\tilde{M}$ with G acting freely on $\tilde{M}$ so that $X = \partial M$, where $M = \tilde{M}/G$. Using this to compute Γ we have

$$N\Gamma(\tilde{X}) = \Gamma(N\tilde{X}) = \int_{\tilde{M}} L_k(p) - d\int_M L_k(p) - \text{Sign } \tilde{M} + d \,\text{Sign } M$$

and since the integrals cancel this gives

$$\Gamma(\tilde{X}) = -\frac{1}{N}\{\text{Sign } \tilde{M} - d\,\text{Sign } M\}. \tag{4.5}$$

Now in [7; §7] an invariant for free G-actions was defined by

$$\sigma(g, \tilde{X}) = -\frac{1}{N}\text{Sign}(g, \tilde{M}) \qquad g \neq 1 \tag{4.6}$$

where $\text{Sign}(g, \tilde{M})$ is computed from the action of g on the middle cohomology of $\tilde{M}$. Since the cohomology of M can be identified with the G-invariant part of the cohomology of $\tilde{M}$ it follows that

$$\frac{1}{d}\left\{\sum_{g \neq 1} \text{Sign}(g, \tilde{M}) + \text{Sign}(\tilde{M})\right\} = \text{Sign } M.$$

Substituting from (4.5) and (4.6) this gives finally

$$\Gamma(\tilde{X}) = -\sum_{g \neq 1} \sigma(g, \tilde{X}). \tag{4.7}$$

Reducing modulo Z this computes $\gamma(\tilde{X})$ in terms of the σ-invariants of [7; §7].

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